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On the Lie–Bäcklund symmetry of linear ordinary differential equations

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Abstract. The symmetry of a second-order linear ordinary differential equation and a system of coupled linear ordinary differential equations is investigated. The equations for characteristic functions, which determine the coordinates of the infinitesimal generators of the Lie–Bäcklund transformations, are found. General solutions of equations for characteristic functions are constructed and the form of infinitesimal generators is obtained.

1. Introduction

We shall analyse the Lie–Bäcklund symmetry of the following two objects

$$y''(x)+V(x)y(x)=0 \tag{1.1}$$

$$\begin{aligned}
&\varphi''(x)+V_{11}(x)\varphi(x)+h_{12}(x)\chi(x)+\ldots+h_{1N}(x)\psi(x)=0\\
&\chi''(x)+V_{22}(x)\chi(x)+h_{21}(x)\varphi(x)+\ldots+h_{2N}(x)\psi(x)=0\\
&\ldots\\
&\psi''(x)+V_{NN}(x)\psi(x)+h_{N1}(x)\varphi(x)+\ldots+h_{NN-1}(x)\omega(x)=0
\end{aligned} \tag{1.2}$$

where $V(x)$, $V_{ii}(x)$, $h_{ij}(x)$ $(i,j=1,\ldots,N)$ are arbitrary functions of an independent variable x. We shall restrict ourselves to the case of a system of two linear differential equations; however, the algorithm of symmetry investigation may be readily extended to a system of N equations.

We are interested in all transformations of independent and dependent variables in (1.1) of the following form

$$\begin{aligned}
x\to\bar{x}&=\bar{x}(x,y,y')=x+\varepsilon\xi(x,y,y')+\ldots\\
y\to\bar{y}&=\bar{y}(x,y,y')=y+\varepsilon\eta(x,y,y')+\ldots
\end{aligned} \tag{1.3}$$

and in (1.2)

$$\begin{aligned}
x\to\bar{x}&=\bar{x}(x,\varphi,\chi,\varphi',\chi')=x+\varepsilon\xi(x,\varphi,\chi,\varphi',\chi')+\ldots\\
\varphi\to\bar{\varphi}&=\bar{\varphi}(x,\varphi,\chi,\varphi',\chi')=\varphi+\varepsilon\eta_1(x,\varphi,\chi,\varphi',\chi')+\ldots\\
\chi\to\bar{\chi}&=\bar{\chi}(x,\varphi,\chi,\varphi',\chi')=\chi+\varepsilon\eta_2(x,\varphi,\chi,\varphi',\chi')+\ldots
\end{aligned} \tag{1.4}$$

with respect to which equations (1.1) and (1.2) are invariant. The second and higher derivatives $y^{(K)}$, $\varphi^{(K)}$, $\chi^{(K)}$ $(K=2,\ldots,\infty)$ are not included in the transformations (1.3) and (1.4) as they may be excluded by means of the original equations (1.1) and (1.2).

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Generally speaking, the set of transformations (1.3) or (1.4) do not form a group but form a monoid, and therefore we shall consider a symmetry monoid and infinitesimal generators of a symmetry monoid. Infinitesimal generators of transformations (1.3) and (1.4) have the form

$$\hat{X} = \xi(x, y, y')\,\partial/\partial x + \eta(x, y, y')\,\partial/\partial y + \dots \tag{1.5}$$

$$\hat{X} = \xi(x, \varphi, \chi, \varphi', \chi')\,\partial/\partial x + \eta_1(x, \varphi, \chi, \varphi', \chi')\,\partial/\partial\varphi + \eta_2(x, \varphi, \chi, \varphi', \chi')\,\partial/\partial\chi + \dots. \tag{1.6}$$

We are sure that the use of the same notations both in the cases of one equation and a system of equations cannot lead to a misunderstanding for they are analysed separately. In what follows we attempt to find the form of infinitesimal generators (1.5) and (1.6) including their first prolongations (Ovsjannikov 1978, Ibragimov 1983).

2. Symmetry of equation (1.1)

Let $p = y'$ and $W(x, y, p) = \eta(x, y, p) - p\xi(x, y, p)$. Using the Lie criterion of invariance of equation (1.1) under the transformations with the infinitesimal generator (1.5) (Lie 1888) and performing tedious calculations, we obtain the next partial differential equation for $W(x, y, p)$:

$$(\partial/\partial x + p\,\partial/\partial y - yV(x)\,\partial/\partial p)^2 W + V(x)W = 0. \tag{2.1}$$

In the special case $W = \eta(x, y) - p\xi(x, y)$, equation (2.1) gives the generators of the eight-parameter group SL(3, R) of point transformations (Ovsjannikov 1978). In the case $V(x) = \text{constant}$, equation (2.1) reduces to the form obtained by Schwarz (1983).

Let $f_1(x)$ and $f_2(x)$ be a pair of fundamental solutions of equation (1.1). In the case of equation (1.1) the Wronskian of the solutions $f_1(x)$ and $f_2(x)$ is equal to unity. Instead of independent variables (x, y, p) we introduce new independent variables (t, α, β):

$$t = x, \qquad \alpha = yf_2'(x) - pf_2(x), \qquad \beta = pf_1(x) - yf_1'(x). \tag{2.2}$$

An important quality of variables α and β is expressed by the following equations

$$\hat{D}\alpha = \hat{D}\beta = 0 \tag{2.3}$$

where

$$\hat{D} = \partial/\partial x + p\,\partial/\partial y - yV(x)\,\partial/\partial p.$$

Taking into account (2.3), equation (2.1) can be written as

$$\partial^2 W(t, \alpha, \beta)/\partial t^2 + V(t)W(t, \alpha, \beta) = 0. \tag{2.4}$$

The general solution of (2.4) is expressed in terms of the fundamental solutions $f_1(t)$ and $f_2(t)$:

$$W(t, \alpha, \beta) = A(\alpha, \beta)f_1(t) + B(\alpha, \beta)f_2(t) \tag{2.5}$$

where $A(\alpha, \beta)$ and $B(\alpha, \beta)$ are arbitrary differentiable functions of two variables. Returning to the infinitesimal operators we obtain the following expression for the first prolongation of (1.5):

$$\hat{X} = A(\alpha, \beta)\,\partial/\partial\alpha + B(\alpha, \beta)\,\partial/\partial\beta. \tag{2.6}$$

The requirement of invariance of differential consequences from (1.1) affects only the structure of the term $\xi(t, \alpha, \beta)\, \partial/\partial t$ in the infinitesimal operator. However, this term may be eliminated by the similarity transformation (Ibragimov 1983). So we do not keep $\xi(t, \alpha, \beta)\, \partial/\partial t$ in (2.6).

In their paper on the symmetry of the second-order linear differential equation Martini and Kersten (1983) have not made the transformation to the canonical set of variables t, α, β in the infinitesimal generator $\hat{X}$ and have not obtained such a compact form of $\hat{X}$ as (2.6), for they have not considered the first prolongation of the infinitesimal generator $\hat{X}$. Thus, the form (2.6) of the infinitesimal generator is in no way dependent on the potential function $V(x)$. The latter defines only the form of canonical variables t, α, β.

3. Symmetry of the system of differential equations (1.2)

Let us introduce the following notation

$$\begin{aligned} &\varphi' = p, \qquad \chi' = q \\ &W_1(x, \varphi, \chi, p, q) = \eta_1 - p\xi, \qquad W_2(x, \varphi, \chi, p, q) = \eta_2 - q\xi. \end{aligned} \tag{3.1}$$

Using the Lie criterion of invariance of the system of equations (1.2) for $N=2$ with respect to the transformation with the infinitesimal generator (1.6), we obtain the following defining equations for the characteristic functions W_1 and W_2

$$\begin{aligned} &[\partial/\partial x + p\,\partial/\partial\varphi + q\,\partial/\partial\chi - (V_{11}\varphi + h_{12}\chi)\,\partial/\partial p - (V_{22}\chi + h_{21}\varphi)\,\partial/\partial q]^2 W_1 \\ &\qquad + V_{11} W_1 + h_{12} W_2 = 0 \\ &[\partial/\partial x + p\,\partial/\partial\varphi + q\,\partial/\partial\chi - (V_{11}\varphi + h_{12}\chi)\,\partial/\partial p - (V_{22}\chi + h_{21}\varphi)\,\partial/\partial q]^2 W_2 \\ &\qquad + V_{22} W_2 + h_{21} W_1 = 0. \end{aligned} \tag{3.2}$$

Let

$$\psi_i = \begin{pmatrix} \varphi_i(x) \\ \chi_i(x) \end{pmatrix}, \qquad i = 1, 2, 3, 4$$

be four vector functions, comprising the set of fundamental solutions of the system (1.2) ($N=2$). As in § 2 we transform the former independent variables x, φ, χ, p, q of equation (3.2) into new independent variables $t, \alpha, \beta, \gamma, \delta$. Set $t = x$. For variables $\alpha, \beta, \gamma, \delta$ we may use the solution of the following system of linear equations:

$$\begin{aligned} &\varphi_1(x)\alpha + \varphi_2(x)\beta + \varphi_3(x)\gamma + \varphi_4(x)\delta = \varphi \\ &\chi_1(x)\alpha + \chi_2(x)\beta + \chi_3(x)\gamma + \chi_4(x)\delta = \chi \\ &\varphi_1'(x)\alpha + \varphi_2'(x)\beta + \varphi_3'(x)\gamma + \varphi_4'(x)\delta = p \\ &\chi_1'(x)\alpha + \chi_2'(x)\beta + \chi_3'(x)\gamma + \chi_4'(x)\delta = q. \end{aligned} \tag{3.3}$$

This system of linear equations is solvable with respect to $\alpha, \beta, \gamma, \delta$, as the determinant of the matrix in the left-hand side of (3.3) is the Wronskian of fundamental solutions, and therefore is always not equal to zero. It is easy to show that the new variables satisfy the following conditions:

$$\hat{D}\alpha = \hat{D}\beta = \hat{D}\gamma = \hat{D}\delta = 0 \tag{3.4}$$

where $\hat{D} = \partial/\partial x + p\,\partial/\partial\varphi + q\,\partial/\partial\chi - (V_{11}\varphi + h_{12}\chi)\,\partial/\partial p - (V_{22}\chi + h_{21}\varphi)\,\partial/\partial q$. Taking into account equations (3.4), the definining equations (3.2) in the new variables $t, \alpha, \beta, \gamma, \delta$ may be transformed to the following form:

$$\begin{aligned} &\frac{\partial^2}{\partial t^2} W_1(t, \alpha, \beta, \gamma, \delta) + V_{11}(t) W_1(t, \ldots, \delta) + h_{12}(t) W_2(t, \ldots, \delta) = 0 \\ &\frac{\partial^2}{\partial t^2} W_2(t, \alpha, \beta, \gamma, \delta) + V_{22}(t) W_2(t, \ldots, \delta) + h_{21}(t) W_1(t, \ldots, \delta) = 0. \end{aligned} \tag{3.5}$$

The system of equations (3.5) differs from the system (1.2) in the parametrical dependence of the functions W_1 and W_2 on $\alpha, \beta, \gamma, \delta$. The general solution of the system (3.5) has the form:

$$\begin{aligned} W_1(t, \alpha, \ldots, \delta) &= A(\alpha, \ldots, \delta)\varphi_1(t) + B(\alpha, \ldots, \delta)\varphi_2(t) \\ &\quad + C(\alpha, \ldots, \delta)\varphi_3(t) + D(\alpha, \ldots, \delta)\varphi_4(t), \\ W_2(t, \alpha, \ldots, \delta) &= A(\alpha, \ldots, \delta)\chi_1(t) + B(\alpha, \ldots, \delta)\chi_2(t) \\ &\quad + C(\alpha, \ldots, \delta)\chi_3(t) + D(\alpha, \ldots, \delta)\chi_4(t) \end{aligned} \tag{3.6}$$

where A, B, C, D are arbitrary differentiable functions of the variables $\alpha, \beta, \gamma, \delta$. For the first prolongation of the infinitesimal generator (1.6) we obtain in terms of variables $\alpha, \beta, \gamma, \delta$:

$$\begin{aligned} \hat{X} &= A(\alpha, \ldots, \delta)\,\partial/\partial\alpha + B(\alpha, \ldots, \delta)\,\partial/\partial\beta \\ &\quad + C(\alpha, \ldots, \delta)\,\partial/\partial\gamma + D(\alpha, \ldots, \delta)\,\partial/\partial\delta. \end{aligned} \tag{3.7}$$

As in § 2 we do not keep the term $\xi(t, \alpha, \ldots, \delta)\,\partial/\partial t$ in (3.7). The form of infinitesimal generator is independent of the potential functions V_{11}, V_{22}, h_{12}, h_{21}, but the form of canonical variables $t, \alpha, \beta, \gamma, \delta$ is fully determined by them. The distinction from the case of one equation consists in a larger degree of arbitrariness in writing down the infinitesimal generator (3.7) (four arbitrary functions of four variables).

4. Conclusion

Canonical variables α, β in the case of one equation and $\alpha, \beta, \gamma, \delta$ for a system of two equations are obviously the 'integrals of motion', defined in terms of dynamical variables. Equations of the form $\hat{D}\alpha = 0$ (see (2.3) and (3.4)) mean the conservation of α along a trajectory in some 'phase' space of the system or the equality to zero of the substantial derivative of α with respect to an independent variable. Thus, the conserved quantities of this type are just the optimal variables for the construction of the first prolongation of infinitesimal generators of the symmetry monoids.

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